The complexity of tropical polynomials and mean payoff games

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Min-plus Semiring

Min-plus semiring (tropical semiring):

\[(T, \oplus, \odot),\]

where \(T\) is \(\mathbb{R}\) or \(\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}\), or \(\mathbb{Z}\), or \(\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}\),

\[x \oplus y = \min\{x, y\},\]

\[x \odot y = x + y.\]
Min-plus Linear Polynomials

Min-plus linear polynomial:

\[ a_1 \odot x_1 \oplus \ldots \oplus a_n \odot x_n \text{ or } \min(a_1 + x_1, \ldots, a_n + x_n). \]

\( x \neq (\infty, \ldots, \infty) \) is a root if the minimum is attained at least twice. That is,

\[ \exists k, l \quad a_k + x_k = a_l + x_l = \min_j (a_j + x_j). \]

This is also called tropical equation.
Example

Consider equation

\[ 0 \odot x \oplus 3 \odot y \oplus 2 \odot z \text{ or } \min(0 + x, 3 + y, 2 + z). \]
Consider equation

$$0 \circ x \oplus 3 \circ y \oplus 2 \circ z \text{ or } \min(0 + x, 3 + y, 2 + z).$$

Solutions: $(0 + t, -3, -2)$, $(0, -3 + t, -2)$, $(0, -3, -2 + t)$ for $t \geq 0$.

Also note that if $(x, y, z)$ is a solution, then $(x + \alpha, y + \alpha, z + \alpha)$ is also a solution.
Min-plus linear equation:

\[ a_1 \odot x_1 \oplus \ldots \oplus a_n \odot x_n = b_1 \odot x_1 \oplus \ldots \oplus b_n \odot x_n \]

or

\[ \min(a_1 + x_1, \ldots, a_n + x_n) = \min(b_1 + x_1, \ldots, b_n + x_n). \]
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Consider

\[ 0 \odot x \oplus 1 \odot y = 2 \odot x \oplus 0 \odot y \text{ or } \]
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\[ 0 \odot x \oplus 1 \odot y = 2 \odot x \oplus 0 \odot y \text{ or} \]

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Solutions: (0, 0).

Here also if \((x, y)\) is a solution, then \((x + \alpha, y + \alpha)\) is also a solution.
Origin

- Min-plus
  Combinatorial optimization, scheduling problems
- Tropical
  Algebraic geometry, mathematical physics.
Algebraic geometry

Consider the algebraic closure of the field of complex rational functions \( \mathbb{C}(t) \). Its elements can be represented by Puiseux series locally at zero:

\[
c_1 t^{d_1} + c_2 t^{d_2} + \ldots,
\]

where \( d_1 < d_2 < \ldots \) are rationals.

The order of the series above is \( d_1 \).

Consider polynomials in \( \mathbb{C}(t)[x_1, \ldots, x_n] \). Then if \((a_1(t), \ldots, a_n(t)) \in \mathbb{C}(t)\) is a solution to some polynomial, then the sequence of orders is a solution to the corresponding tropical equation.
We consider systems of tropical linear equations

\[ \min_{1 \leq j \leq n} \{ a_{ij} + x_j \}, \; 1 \leq i \leq m, \]

We call the set of common roots of polynomials of the system by min-plus linear prevariety.
We also consider systems of min-plus linear equations

\[ \min_{1 \leq j \leq n} \{ a_{ij} + x_j \} = \min_{1 \leq j \leq n} \{ b_{ij} + x_j \}, \; 1 \leq i \leq m. \]
The Main Problem

We are mostly interested in solving systems of tropical linear equations and systems of min-plus linear equations.

In the classical case there are polynomial time algorithms for this problem.

In min-plus case there are no polynomial time algorithms known.

It is however known that the solvability problem is in the complexity class $\text{NP} \cap \text{coNP}$. 
Problems in $\text{NP} \cap \text{coNP}$

- Linear programming
Problems in \( \text{NP} \cap \text{coNP} \)

- Linear programming known to be in \( \text{P} \)
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  - parity games, mean payoff games,
  - stochastic games
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~ Bezem et al. (2010)
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Bezem et al. (2010)
Grigoriev, P. (2012)
Mean Payoff Games

Ehrenfeucht, Mycielski (’79); Gurvich, Karzanov, Khachiyan (’88).
Two players, Alice and Bob, move a token over bipartite graph.

Alice tries to maximize the sum of edge labels, Bob tries to minimize it.

Game: $v_1, v_2, v_3, \ldots$

Value of the game: $\limsup_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} \rho(v_i, v_{i+1})$. 
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2
\[ \rightarrow \]
-3

0

Alice

-1

0

1

2

-1

Bob

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-3
0
-1
1
2
0 + 0 + 2

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![Bipartite Graph Diagram]

Value of the game: \( \limsup_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} \rho(v_i, v_{i+1}) \).

Alice wins if the value is positive. Otherwise Bob wins.

It is known that there are optimal positional strategies.
Mean Payoff Games

Mean payoff games problem: given a labeled graph decide whether Alice has a winning strategy.

The problem is in $\text{NP}$. For certificate we can take a winning strategy for Alice.

The problem is also in $\text{coNP}$. For certificate we can take a winning strategy for Bob.
Problems We Consider

**Solvability problem TropSolv**: Given an integer matrix $A \in \mathbb{Z}^{m \times n}$ decide whether the corresponding tropical linear system is solvable.

**Equivalence problem TropEquiv**: Given two integer matrices $A \in \mathbb{Z}^{m \times n}$ and $B \in \mathbb{Z}^{k \times n}$ decide whether the corresponding min-plus linear prevarieties are equal.

**Dimension problem TropDim**: Given an integer matrix $A \in \mathbb{Z}^{m \times n}$ and a number $k \in \mathbb{N}$ decide whether the dimension of the min-plus prevariety is at least $k$.

Note that the min-plus prevariety in $\mathbb{R}^n$ is a finite set of polytopes.

We also consider analogous problems over $\mathbb{Z}_{\infty}$ and also analogous problems for min-plus linear systems.

All these problems are polynomial time solvable in classical case.
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## Results

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All results are true both above $\mathbb{Z}$ and $\mathbb{Z}_{\infty}$.

Note that all problems in the first two columns are polynomial time equivalent.
We can consider the systems of linear min-plus inequalities. But this is the same as equations.
Min-plus inequalities

We can consider the systems of linear min-plus inequalities. But this is the same as equations.

\[ L_1(x) = L_2(x) \text{ iff } L_1(x) \geq L_2(x) \text{ and } L_1(x) \leq L_2(x). \]

\[ L_1(x) \leq L_2(x) \text{ iff } L_1(x) = \min(L_1(x), L_2(x)). \]
Min-plus and mean payoff games

Alice

Bob

\[
A = \begin{pmatrix}
    r_1 & c_1 & c_2 & c_3 \\
    r_2 & -2 & 0 & \infty \\
    r_3 & \infty & \infty & -1 \\
    \infty & \infty & \infty & \infty
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
    r_1 & c_1 & c_2 & c_3 \\
    r_2 & -3 & \infty & \infty \\
    r_3 & \infty & -1 & 0 \\
    \infty & \infty & \infty & -1
\end{pmatrix}
\]

\[
a_{ij} = -\alpha, \quad b_{ij} = \beta
\]
Min-plus and mean payoff games

Alice wins iff there is a solution to the system.
Theorem
\( \text{TropSolv} \) is polynomial time equivalent to mean payoff games.

We will show that \( \text{TropSolv} \) is polynomial time equivalent to the solvability problem for the systems of min-plus equations.
Tropical Solvability

Theorem

\text{TropSolv} is polynomial time equivalent to mean payoff games.

We will show that \text{TropSolv} is polynomial time equivalent to the solvability problem for the systems of min-plus equations.

For this we give reductions in both directions.
Solvability: Tropical $\rightarrow$ Min-plus

The following more strong connection is actually true.

**Lemma**

*For a given system of linear tropical polynomials we can effectively construct an equivalent system of linear min-plus polynomials.*
Solvability: Tropical $\rightarrow$ Min-plus

\[
\min\{y_1, \ldots, y_n\} \text{ is attained at least twice }
\]

\[
\text{iff }
\]

For all \( i \in \{1, \ldots, n\} \)
\[
\min\{y_1, \ldots, y_n\} = \min\{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n\}
\]

\[
\text{iff }
\]

For all \( i \in \{1, \ldots, n\} \)
\[
\min\{y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n\} =
\min\{y_1, \ldots, y_{i-1}, y_i + 1, y_{i+1}, \ldots, y_n\}
\]
Solvability: Tropical → Min-plus

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\min\{y_1, \ldots, y_n\} \text{ is attained at least twice}
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\text{For all } i \in \{1, \ldots, n\}
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\[
\text{iff}
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\text{For all } i \in \{1, \ldots, n\}
\min\{y_1, \ldots, y_i-1, y_i, y_i+1, \ldots, y_n\} =
\min\{y_1, \ldots, y_i-1, y_i + 1, y_i+1, \ldots, y_n\}
\]

Let \( y_i = x_i + a_i \).
Solvability: Min-plus → Tropical

In the other direction we do not have such a tight connection. But we have it if we do not look at the neighborhood of infinity.

We say that two sets $S$ and $T$ in $\mathbb{R}^n$ are $C$-equal for an integer $C$, if $S \cap B(0, C) = T \cap B(0, C)$, where by $B(0, C)$ we denote the ball centered in 0 with the radius $C$.

Lemma

For any min-plus linear system $A \odot x \leq B \odot x$ over $n$ variables and for arbitrary integer $C$ there is a tropical linear system $D$ over $2n$ variables and an injective linear transformation $H: \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ such that all solution of $D$ lie in $\text{Im}(H)$ and the image of the solutions of $(A, B)$ and the set of solutions of $D$ are $C$-equal.
Lemma

Let $k \leq n$ and consider arbitrary vector $\vec{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$. Then for any $C \in \mathbb{Z}$ there is a tropical linear system $A \in \mathbb{Z}^{m \times n}$

$$A = \begin{pmatrix} \vec{a} \\ \vec{a} \\ \vec{a} \end{pmatrix} \begin{array}{c} \geq C \end{array},$$

where $m = n - k + 1$, such that for any solution of $A$ and for any row the minimum is attained at least twice in the $\vec{a}$-part of the row.
Technical Lemma

Proof sketch.

\[ C = 99 \]

\[
\begin{pmatrix}
\vec{a} & 100 & 100 & 100 \\
\vec{a} & 99 & 100 & 100 \\
\vec{a} & 100 & 99 & 100 \\
\vec{a} & 100 & 100 & 99 \\
\end{pmatrix}
\]
Solvability: Min-plus $\rightarrow$ Tropical

Lemma
For any min-plus linear system $A \odot x \leq B \odot x$ over $n$ variables and for arbitrary integer $C$ there is a tropical linear system $D$ over $2n$ variables and an injective linear transformation $H : \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ such that all solution of $D$ lie in $\text{Im}(H)$ and the image of the solutions of $(A, B)$ and the set of solutions of $D$ are $C$-equal.

Proof:
For each variable $x_i$ of $(A, B)$ we have two variables $x_i$ and $x_i'$ of $D$. For each $i$ we apply Technical Lemma with $\vec{a} = (0, 0)$, $C = C$ to the variables $x_i, x_i'$. Denote the resulting system by $T_i$. In each its solution the variables $x_i$ and $x_i'$ are equal. We include systems $T_i$ for all $i$ into the system $D$. 
Solvability: Min-plus $\rightarrow$ Tropical

Assuming $x_i = x'_i$ for all $i$ we have

$$\min(a_1 + x_1, \ldots, a_n + x_n) \leq \min(b_1 + x_1, \ldots, b_n + x_n)$$

iff for all $i = 1, \ldots, n$

$$\min(a_1 + x_1, \ldots, a_n + x_n) \leq b_i + x_i$$

iff for all $i = 1, \ldots, n$

$$\min(a_1 + x_1, a_1 + x'_1, \ldots, a_n + x_n, a_n + x'_n, b_i + x_i)$$

is attained at least twice.

\[\square\]

Even more strong connection is true over $\mathbb{Z}_\infty$. 

Thus we have that tropical linear systems and min-plus linear systems are in some sense “equivalent”.

This equivalence is enough to prove that
1. the solvability problems for these systems are equivalent;
2. the equivalence problems for these systems are equivalent;
3. the dimensional problems for these systems are equivalent.

Note that our “equivalence” shows that geometrical structure of tropical linear systems and min-plus linear systems are almost the same.
## Results

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We move to the discussion of \textsc{NP}-completeness of \textsc{TropDim}.

**Definition**
Let $A$ be a matrix of size $m \times n$. We associate with it the table $A^*$ of the same size $m \times n$ in which we put the star $\ast$ to the entry $(i, j)$ iff $a_{ij} = \min_k \{a_{ik}\}$ and we leave all other entries empty.

Note that $x = (x_1, \ldots, x_n)$ is a solution to the system $A$ iff there are at least two stars in every row of the table $(\{a_{ij} + x_j\}_{ij})^\ast$.

Below we assume that in all tables we consider there are two stars in each row.
Definition

The block triangular form of size $d$ of the matrix $A$ is a partition of the set of rows of $A$ into sets $R_1, R_2, \ldots, R_d$ (some of the sets $R_i$ might be empty) and a partition of the set of columns of $A$ into nonempty sets $C_1, \ldots, C_d$ with the following properties:

1. for every $i$ each row in $R_i$ has at least two stars in columns $C_i$ in $A^*$;
2. if $1 \leq i < j \leq d$ then rows in $R_i$ have no stars in columns $C_j$ in $A^*$. 
Dimension: Block Triangular Form

\[ A^* = \begin{pmatrix}
  & C_1 & C_2 & \cdots & C_d \\
  R_1 & \begin{pmatrix}
    * & * \\
    * & * & \emptyset & \emptyset & \emptyset \\
  \end{pmatrix} \\
  R_2 & \begin{pmatrix}
    * & * & * & \emptyset & \emptyset \\
    * & * & * & \emptyset & \emptyset \\
    * & * & * & \emptyset & \emptyset \\
  \end{pmatrix} \\
  \vdots & \begin{pmatrix}
    * & * & * & \emptyset & \emptyset \\
    * & * & * & \emptyset & \emptyset \\
    * & * & * & \emptyset & \emptyset \\
  \end{pmatrix} \\
  R_d & \begin{pmatrix}
    * & * & * & * & * & \emptyset & \emptyset \\
    * & * & * & * & * & \emptyset & \emptyset \\
    * & * & * & * & * & \emptyset & \emptyset \\
  \end{pmatrix}
\end{pmatrix} \]
Theorem

For a solution $x$ of the tropical linear system $A$ the local dimension of the system $A$ in point $x$ is equal to the maximal $d$ such that there is a block triangular form of the matrix $\{a_{ij} + x_j\}_{ij}$ of size $d$. 
Min-plus Dimension is NP-complete

**Theorem**

\(\text{TropDim is NP-complete (both over } \mathbb{Z} \text{ and } \mathbb{Z}_\infty).\)

To show the containment in NP we can give as a certificate the point in which the dimension is achieved and the block triangular form.

To show the completeness we give a reduction from the vertex cover problem \(\text{VertexCover}:\) given a graph \(G\) and an integer \(k\) decide whether there is a set \(S\) of vertices of size at most \(k\) such that for each edge it least on of its ends is in \(S\).
Higher degree

Min-plus monomial

\[ M(x) = d_1 x_1 + \ldots + d_n x_n, \]

where \( d_i \geq 0 \), \( d_i \) are integer.

Min-plus polynomial

\[ p(x) = \min_i M_i(x), \]

where \( M_i \) are tropical monomials.

We consider min-plus polynomial equations of the form

\[ p(x) = q(x). \]
Nullstellensatz

Theorem (Classical Nullstellensatz)

The system of polynomials $f_1, \ldots, f_m$ over algebraically closed field does not have a solution iff 1 lies in the ideal generated by $f_1, \ldots, f_m$.

Naive tropical reformulation is not true.

The min-plus system

$$x = 0, \quad x = 1$$

has no solution, but cannot generate $1 = 0$. 
Min-plus Nullstellensatz

Theorem
The system of min-plus polynomial equations $f_1 = g_1, \ldots, f_m = g_m$ has no solution iff we can construct an algebraic combination $f = g$ of them such that for each monomial $M = x_1^{\odot d_1} \odot \ldots \odot x_n^{\odot d_n}$ its coefficient in $f$ is greater than its coefficient in $g$.

There is also an analogous result for the tropical case.